THE PURE ECONOMICS OF THE COASE THEOREM

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The selection of Ronald Coase as recipient of the 1991 Nobel Prize in economics will encounter a wide range of response from members of the economics profession. That the choice will be highly controversial is an understatement. More importantly, even among those who would support Coase, there will be much discussion and disagreement as to why Coase “deserved” this highest of honors. Unlike Nobel laureates in pure science or medicine, those in economics tend to be rewarded for the totality of career contributions rather than for any particular major discovery. Whether Coase’s proposition is or is not a theorem, its impact has been like that of a major discovery. The 1960 “social cost” article may well be the all-time leader in the citations department and it is reasonably accurate to assert that the modern field of law and economics began with that paper. The 1967 “nature of the firm” paper also has an enormous number of citations, but was largely ignored prior to 1960.

However important or insightful Coase’s many other writings may be, they all tend to embody the analytical point of view that underlies the two classic articles. If Coase has received the Nobel Prize for the totality of his career work, as I believe to be the case, it is important for economic and legal scholars alike to understand exactly what Coase has contributed. Recently, Coase [1988, 1-7] complained that his analytical point of view, though the focus of considerable attention, is neither accepted nor understood by economists. He also states that legal scholars have accepted his views, but stops short of saying that legal scholars have understood them. There is no better evidence of economists’ lack of understanding than the so-called mathematical proofs of the invalidity of the Coase Theorem. The primary objective of this paper is to isolate and explain Coase’s contributions to pure economic theory. The secondary objective is to expose the nonsense underlying the mathematical proofs and to repair the damage. What emerges is a pristine version of Coase’s loosely stated proposition which, despite apparent formal differences, lends broad support to the more common interpretations (of Coase) found in the literature.

To accomplish these objectives, it is necessary to piece together a number of issues and concepts which, though fundamental to “intermediate” price theory, tend to be taken for granted both in the telling of Coasian “stories” and in the derivation of formal mathematical theorems. The focal point is the standard assumption of technical efficiency which (1) underlies production functions, (2) underlies cost functions, and (3) defines the representative firm in models of pure competition. Those who allege to have disproved the Coase Theorem fail to think through the logic that precedes the setup of their models. The irony is that Coase has been asking economists to think about such things for over half a century.

Section I contains a brief summary of the conditions underlying a “long-run”, perfectly competitive equilibrium, concluding with a clear statement of the important economic considerations that are bypassed through the assumption of technical efficiency and how the meaning of technical efficiency changes in the context of so-called externality problems. In Section III, I will relate Coase’s [1937] discussion of the nature of the firm directly to the substance of the 1920s cost controversy and the concept of
technical efficiency. The "missing link" here is the manner in which the firm's administrative decisions supercede the price mechanism. Finally, Sections IV and V address the legal-economic issues underlying Coase [1960], together with a rigorous translation of the issues into a symbolic framework that is suitable for mathematical analysis.

I. ELEMENTS OF PERFECT COMPETITION

In textbook analyses of a purely competitive industry, the core of the model is the "representative" firm (henceforth, the firm), acting as price taker. The partial equilibrium story includes profit-maximizing level of output, with zero economic profit through "entry/exit", cost-minimizing input combination and implicit constant returns to scale in production through the product exhaustion theorem. The general equilibrium version requires simultaneous equilibrium of product and factor markets. Underlying both partial and general equilibrium versions is the presumption that firms administer their resources "properly". In this section, I will lay out the mathematical mechanics of each step and follow up with a discussion of its relevance to Coase. The larger theme is that the elements of these four analytical "boxes" must be in concert with one another. Notation, which is largely conventional, is summarized in the appendix.

Output, Cost, and Firm Size

Given a cost function \(c(x)\), the firm adjusts its output to a level \(x^*\) so that \(c(x^*) = p\), or "price equals marginal cost". The market price is exogenous to the firm, but for the industry, there is a demand curve, \(p(x)\) or \(p(x)\), where aggregate output \(x\) must equal \(n\), and in the long-run, the firm will adjust to "eliminate economic profit". The model thus yields a specific mathematical solution for both \(x\) and \(n\), as well as an inferential solution for \(p(x)\):

\[
\begin{align*}
c(x^*) &= p(n) \\
c(x^*)/n &= p/n^*\end{align*}
\]

By inference from the two conditions, \(n = n^*\) and, therefore, all firms operate at a point of minimum long-run average cost. The efficiency properties of this position are obvious. The larger issue is that if the allocative failure of this condition which the critics cite as the failure of the Coase Theorem. Yet, the analysis is not carried beyond the simplistic "entry and exit" story.

Mathematically, this condition follows from the existence of unique solutions for \(x\) and \(n\). Economic interpretation is another matter. The model raises the deductive question of why the "long-run ac curve" must be \(u\)-shaped. Part of the answer lies in implicit adjustments in "plant size", the colloquial synonym for the fixed inputs (e.g., "generic capital") embodied in the "short-run" version of the model. The complete answer, however, requires an examination of the relationship between output and inputs and implicit adjustments in all prices.

The technical economic interpretation of minimizing long-run average cost is that the firm operates at a point of constant returns to scale (henceforth, CRS). That is, potential economies of scale are fully exhausted. The classic example of this, Smith's pin factory, provides ample detail of what firms do to accomplish this feat, but the details are obscured by the generality of the model. Underlying the model are the two sub-
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choice of output, $\alpha$ must equal output price $p$. Ceteris paribus, the quantity of each input is adjusted so that value of marginal product (vmp) is equal to price:

$$vmp = p \cdot \alpha(p) = w_i$$

By substitution of these conditions back into the definition of total costs ($c = w\alpha$), we get

$$c(x, \alpha) = \sum_{i=1}^{n} w_i \alpha_i \alpha(p) = p \sum_{i=1}^{n} \alpha_i \alpha(p) \cdot \alpha, \quad i = 1, 2, \ldots, n$$

The zero-profit condition of $px = c(x, \alpha)$ thus reduces to

$$x = \sum_{i=1}^{n} \alpha_i \alpha(p) \cdot \alpha, \quad \text{since the p on both sides of the equation cancels out.} \quad \text{(2.1)}$$

Since all pecuniary market prices have been removed, the latter is purely technological and implies that the production function exhibits CRS at the specific point $\alpha$. This result, known as the "product exhaustion theorem," does not require that CRS apply throughout the entire range of the production function. Quite the contrary, the existence of a unique solution for $x$ and $\alpha$ requires increasing returns at lower rates of $x$ and decreasing returns at higher rates of $\alpha$. My purpose in going through this rather well-known exercise is to shift later discussions of Coase to "common" ground (meaning real rather than pecuniary terms). Specifically, the "bargain" lastly to be specified is a pure exchange. That "exchange" pertains to decisions which are not internal to the operation of firms -- i.e., to matters of technical, rather than economic, efficiency. Coase does make these distinctions. However, they are almost unnoticeable, perhaps for obvious reasons. In principle, the $m$-input case is no different from the case of two generic (labor and capital) inputs or that of all purpose "putty" input. To Coase, this means that production functions (like utility functions) are "ornaments." The choice of variables determines the form rather than substance of economic cost.

Technical Efficiency

The foregoing mechanics are elementary to economists and may be found even in intermediate theory textbooks and in many undergraduate curricula. There are three key elements. First, the production function is defined in such a way that all of its variables have competitive market prices. Second, in equilibrium, all of the pecuniary variables reduce to "real" phenomena, meaning the curvature of the production function at the point of equilibrium. Third, the model per se embodies no assumptions regarding how the firms or the market "adjust" from disequilibrium to equilibrium positions. The operative phrase here is that rational expectations about price adjustments are fully internalized into the cost functions. These principles provide the foundation for modern general equilibrium theory. Pure economic theorems (particularly those concerned with the efficiency of the system as a whole) relate to economies which operate without money and firms which operate without entrepreneurship or management. Analytically, technical efficiency provides the "perfect substitute" for entrepreneurship and management. Technical efficiency embodies the presumption that any numerical combination of inputs are organized in such a way to maximize output. To modern economic theorists, the organizational problem is either irrelevant (because the firm is presumed not to exist as a matter of course) or too complex to be modeled in any general way. As a transition to the following section, I offer two simple observations. First, in the context of the preceding discussion, suppose an imaginary firm produces any steel from some quantity of "putty." Technical efficiency requires the firm to mold the putty into $x$; inputs in the process of producing the steel. The "prices" of the putty may be defined as unity, but there must be $m$ implicit prices lurie the production function. Second, and more importantly, production functions necessarily abstract from spatial and temporal dimensions. These dimensions, which are obviously important to Coase, are fundamental to any legal definition of property rights and to the economic concept of an externality. One need only think about the movement and control of cattle to see the clear relevance of these dimensions. What is not obvious is that under conditions of externality, rational firms do not pursue technical efficiency in the strict sense.

The purpose of the following section is to clarify this proposition by means of a simple economic example. The proposition will be applied to Coase in subsequent sections.

II. A RECONSIDERATION OF TECHNICAL EFFICIENCY

The point of the pin factory is that large-scale production enables a firm to realize real economies through specialization and the division of labor. Ranges of increasing returns to scale are built into mathematical production functions only to be exhausted in equilibrium by the rationality of the firm and the competitive forces of the market. The division of labor is also built into the production function, but its presence is too obscure to be noticed. Suppose, for example, that the production function is $x(\alpha, q)$, where $\alpha$ now represents a vector of all other inputs and $q$ denotes some quantity of generic labor. Implicitly, $q$ must be allocated to a set of distinguishable tasks, akin to the long list of things that are done to raw metal to produce a pin. A priori, there is no way to know what the tasks are or how many tasks are performed; so, let us just suppose that there are two of them. If division of labor matters, then the production function expands to $x(\alpha, q_1, q_2)$. For any given quantity of labor, the organizational problem is to assign workers (or worker-time) to tasks in such a way to maximize $x$. The formal problem is:

$$x(\alpha, q) = \max_{x(\alpha, q_1, q_2)} \quad \text{subject to the constraint } q = q_1 + q_2$$

The mathematical solution is to equate marginal productivities of each task. Note, in particular, the analogy between the constrained maximization problem underlying the definition of the production function and the constrained minimization problem underlying the definition of the cost function. Given the solution to this problem, the (singular) marginal product of labor is implied by the problem statement. The $mp$ of each task are implicit prices. Moreover, if spatial and temporal dimensions were introduced, the mathematical exercise and the infinity of implicit prices could be carried on well beyond the point of absurdity. Nevertheless, this is exactly what the organizational firm is presumed to be able to accomplish. The operative phrase here is that rational expectations about spatial-temporal matters imply an infinity of implicit prices that are fully internalized into the definition of the production function. Since we cannot generally know what the specific tasks are, it may be that some are more dangerous than others. If we extend the definition of homogeneous labor
to include identical preferences for risk, the final arrangement will determine not only a rate of output, but also an expected number of accidents. If we define \( n \) as expected number of accidents, the full problem expands to three equations:

\[
\begin{align*}
(1) & \quad x = x(0, q, q) \\
(2) & \quad s = s(0, q, q) \\
(3) & \quad q = q + q.
\end{align*}
\]

The three equations reduce to one by substituting (2) and (3) into (1), thereby eliminating two variables from the problem statement. Specifically, we get \( x = x(0, q, q) \) but \( q \) and \( q \) disappear. Given any \( s \) and \( q \), the firm chooses \( x \) by allocating \( q \) between "risky" and "riskless" tasks. Implicitly, the firm trades output for accidents. Put another way, the number of accidents is an implicit input in the production function. If the firm maximizes output with respect to \( q \) and \( q \), it is implicitly maximizing with respect to \( s \). That is:

\[
x(0, q) = \max_{s} x(0, q, s) = \max_{s} x(0, q, q) \quad s.t. \quad q = q + q.
\]

If the firm does not have to pay for accidents, it will act as if \( x(0, q) \) is its production function. However, if market wage rates, in the "long-run," include an appropriate risk premium, the firm will use \( x(0, q, q) \), not \( x(0, q) \). If the firm is required to compensate accident victims, the firm will act as if \( x(0, q, s) \) is its production function, but market wages will not include the risk premium. Either way, the long-run cost function is the same, provided that the analysis includes a careful accounting of implicit price adjustments. The only difference is that, under one arrangement (no direct responsibility for accidents), the firm appears to behave in a socially irresponsible manner. The incentive structures appear to be different because they have different administrative forms. Whether or not this scenario is "plausible" is beside the point. The exercise is offered not as a theory of industrial accidents, but as an example of the kind of "shell game" that can be played with economic margins. In particular, it was designed to provide some inferential perspectives into what was bothering Coase in 1957 and why economists did not pay any attention to his theory of the firm. By 1960, this economic "shell game" was firmly intact and Coase was ready to put that game into reverse gear with regard to the farmers and ranchers. There is a remarkable consistency between Coase [1957] and Coase [1960]. The connection between the two papers is the subject of the following section.

III. ADMINISTRATION AND THE COASIAN FIRM

The so-called cost controversy of the 1920s concerned the extent to which the logic of competition ought to be carried sensibly. These considerations prompted Sraffa’s discussion leading up to the following passage:

Business men, who regard themselves as being subject to competitive conditions, would consider absurd the assertion that the limits to their production is to be found in the internal conditions of production in their firm, which do not permit of the production of a greater quantity without an increase in cost. The chief obstacle against which they have to contend when they want gradually to increase their production does not lie in the cost of production—which, indeed, generally favours them in that direction—but in the difficulty of selling the larger quantity of goods without reducing the price, or without having to face increased marketing expenses...[1926, 543]

Much of the framework for Coase’s discussion of the nature of the firm may be inferred from this statement. Nevertheless, there are significant differences in both emphasis and substance. Those differences were largely overlooked in 1957 because the profession had just spent the better part of two decades talking around the issues. That period marked a transition between the texture of Marshall (which emphasized time dimensions) and the rigid logic of perfect competition, in which all past decisions (through “long-run” adjustments) and expectations about future decisions (through perfect information) are collapsed to a single point in time.

Despite the "lip-service" which Sraffa paid to one aspect of "transactions costs" (marketing expenses), his emphasis was upon the internal conditions of production, which were in substance separable from (not interdependent with) the problems of transacting. Coase’s emphasis upon transactions costs is but a superficial difference, largely attributable to the professor’s emphasis upon the conditions of production. The substance is that the internal conditions of production are fraught with transactions costs of a different kind, and the organizational firm must trade-off one for another. In the narrow context of a single-product firm, this trade-off is between the achievement of technical efficiency (an internal matter) and the pursuit of favorable terms of exchange (a matter of external interaction with "the market"). The organizational firm must plan such activities, and that plan determines a kind of equilibrium position for the firm. In any practical sense, however, these activities are responses to disequilibrium phenomena and the product of differences, rather than similarities, among organizations with the same general objectives.

Although he did not dwell on the transactions costs that pertained to the internal conditions of production, Coase made note of them through the following example:

(On economic theory we find that the allocation of factors of production between different uses is determined by the price mechanism. The price of factor A becomes higher in X than in Y. As a result, A moves from Y to X until the difference between the prices in X and Y, except in so far as it compensates for other differential advantages, disappears. Yet in the real world, we find that there are many areas where this does not apply. If a worker moves from department Y to department X, he does not go because of a change in relative prices, but because he is ordered to do so. [1937, 385])

The movement from department Y to department X is a (very minor) disequilibrium adjustment. Following our discussion of technical efficiency in the preceding section, it is clear that the insertion of a "y" for labor in the production function presupposes that the firm can work out in advance where workers are to be stationed. It is also presumed that workers obey orders and that the movement from Y to X is without cost. Yet the issuance of an order, the response time to that order, and any other subsidiary associated
with it all embody a type of transactions cost. These costs tend to be individually negligible, but collectively significant. The simple admission that the firm makes administrative decisions raises such ethical questions as (1) which "variables" within the production function have "market" prices and which do not; (2) at what "point" does administration supersede the price mechanism; and (3) how, and in what sense, is that point determined? Coase's general answer to (3) is that there are trade-offs between two imperfect systems, where the mechanics of one depends upon the mechanics of the other. With regard to the operation of the economic system as a whole, there is in principle a "network" of similar trade-offs at work. Whatever we might choose to define as a particular market operates around a set of established rules or conventions which, at any point in time, have evolved to fit the set of problems confronted by the participants.

Precisely because there are no precise answers, economic solutions based on (marginal) optimality conditions must be precisely wrong. Of itself, this is an empty proposition. However, Coase's criticism of economics is far more fundamental and reduces to the following: (1) The use of economic models to define an imperfection in an otherwise perfect system and to derive corresponding optimal solutions is a logically inconsistent exercise; (2) Furthermore, the use of such models as guidelines for the formulation of satisfactory policy embodies a bias through the "mind set" that is conditioned by the structure of the models.

Prior to 1960, Coase's academic reputation was built around this theme. He consistently lent important perspectives to policy discussions, but his work was not regarded as a fundamental contribution to the logic of pure economic theory. In many circles, the Coase Theorem is still not regarded as pure theory because it seems to defy formal mathematical proof [Cooper, 1987]. Many mathematical exercises suggest the proposition to be a logical impossibility. In particular, the editors [1977] of the Journal of Economic Theory proclaimed a paper by Starrett [1972] to have destroyed its validity. In the following section, I will show that these results flow from internally inconsistent assumptions and explain the elements of a mathematical Coase Theorem.

IV. COASE'S PROPOSITION AS A PURE ECONOMIC THEOREM

What is missing from all mathematical models of Coase is the all-important set of administrative decisions underlying the technical efficiency assumption embodied in the definition of production functions. The models proceed from poorly defined production functions or from cost functions which bypass the usual mathematical derivation [Tybout, 1972; Schulshe and Argo, 1974; Fisch, 1979; Hamilton et al., 1989]. One way or another, the technology is reduced to two cost functions, a c(x) for a "culprit" firm (e.g., Coase's "rancher") and a c(y, x) for a "victim" firm (e.g., Coase's "farmer"), where x and y denote outputs for the respective firms. Assuming a small group of neighboring firms can bargain to a solution which maximizes joint profit with respect to x and y, an insurmountable problem arises with regard to the long-run zero-profit condition. Specifically, if "entry and/or exit" drives joint profits to zero, it is impossible for both groups of firms to minimize average costs (to operate at a point of CRS) unless some mechanism (tax or liability rule) forces culprits to pay for damages at the margin. Without holding culprits "liable for damages", it therefore appears that there will be "too many" culprits and "too few" victims. In relation to the efficient position (achieved by the liability rule), dropping the liability rule effectively forces victims to "knee" culprits, causing more
culpits to enter and some victim firms to exit. As in the single-industry model, "entry and exit" causes further price adjustments which, in this case, causes the efficient scale conditions to be violated.

The Modeling Issues

The mathematics are, of course, correct. However, the models are set up in a way that ignores virtually all of Coase's criticisms of the neoclassical firm and its production function and cost function. The fundamental issues are all embodied in Coase's farmer and rancher's paradigm, but the models bypass them as though they did not exist. First, acceptance of even the possibility of a bargain between two competitive firms allows people into the model. Second, acceptance of the fact that crops are trampled by cattle admits a spatial dimension. Third, the story gives people something to do after inputs have been acquired from the market and before final outputs are sold. Accurate modeling requires that these features be compressed into the problem statement. In particular, production functions must be modified in a way that reflects the complexity of the problem, and the connections between cost, production, and technical efficiency must be clearly established. Models used to discredit Coase reflect the attempt to do this. Consequentially, the cost functions embody the presumption that the "culprits" act as though they have no incentive to choose production processes that are less harmful. In effect, the presumption is that bargains either do not occur or are not expected to occur. The inconsistency here is clear, and the theme will be explained in more detail below.

The Story

To untangle the specific elements of the story, let us first consider the implicit sequence of events that are compressed into a point in time:

1. Property rights are clearly defined.
2. "Agents" become farms (farmers or ranchers) by acquiring productive inputs.
3. Neighboring farmers and ranchers "bargain".
4. Final production plans are implemented.
5. Final outputs are sold.

Note, in particular, that there are no farmers or ranchers until after property rights are clarified. An agent becomes a firm by acquiring control over productive inputs, but how does a "firm" become a "farmer" or a "rancher"? Does the agent also buy a production function? Or do the inputs acquired by the agents have certain characteristics which enable us to infer that one type is a farmer and the other type is a rancher? In the so-called real world, the answer is "yes", but in the theoretical world of generic or "all-purpose" inputs (a), the answer is "no". In short, it is impossible to discuss this problem without attaching labels to inputs.

Although we cannot know exactly what the specific inputs are, generic classification is possible. For example, both types of farms use land (a) and labor (d), but each also uses inputs that are unique to the choice of product, say, livestock (a) and seed (b).
The specific list is a bit too crude, but at least we have defined what the inputs are: 

\[
v = (v, q, a, b)
\]

Let us return to step (1). In order to define property rights as they apply to neighboring farmers and ranchers, the law must first define the terms, "neighbor," "farmer," and "rancher." Now that \( v \) has some substance, it is possible to say that (1) a farmer is an agent who acquires inputs \( q, a, b \) for the purpose of producing output \( y \); (2) a rancher is an agent who acquires inputs \( q, a \) for the purpose of producing output \( z \); and (3) the two agents are neighbors if their "z"s have common boundaries. Superficially, the property rights concern the extent to which a rancher's "z" input is physically present on a farmer's "y" input.11 If some "straying" is inevitable, a portion \( \alpha \) of the rancher's herd \( \alpha \) will walk on "cultivated" land. In order to define these latter terms, it is necessary to think about technical efficiency and the production functions.

If the land input is a homogeneous commodity, there is no distinction between cultivated land and grazing land when agents acquire land in step (2). Similarly, when rancher agents acquire cattle, there is no distinction between straying cattle and the remainder of the herd. Actual outcomes are matters of choice and relate directly to how the land inputs \( q \) input is utilized. The input may be used to cultivate land (plant, tend, harvest crops) or to control cattle (supervise cattle on noncultivated land and/or police the economic boundary). These choices may be administered unilaterally by independent agents or jointly by mutual agreement. If they are implemented unilaterally, the economic and legal boundaries are one and the same, but if they are implemented by mutual agreement, the two diverge. In essence, the property rights and the implied liability rule are superceded by the mutual agreement. To understand this implication, it is necessary to move on to step (3), the "bargain."

The "Bargain"

The full mathematical mechanics (DeSerpa, 1991a-b) of this exercise is beyond the scope of this paper, but I will appeal to the form of the earlier discussion of technical efficiency. It is axiomatic that a bargain requires a mutually acceptable change from an initial position to a final position. In context, the bargain cannot change prices, either the prices already paid for the inputs or those that are to be received for the outputs. Furthermore, the bargain cannot change the quantities of the inputs that were acquired independently by the firms. The initial positions are defined by the maximum outputs that each firm could produce, given their prior input purchases and the rights and duties established by the law. The two extreme legal positions are "no trespassing" (\( s = 0 \)) and "crescere agricolare" (no legal limitation on \( s \)). In order to understand the subtleties of the bargain and how the law ultimately affects it, it is critical to focus on the culp-rancher's production alternatives, holding constant the quantities of all inputs.

Specifically, given the rancher's choice of the market inputs, \((a, q, o)\), there is an administrative trade-off between output \((o)\) and straying cattle \((\alpha)\). This trade-off depends on the manner in which the control input \((o)\) is actually utilized. The more attention that is paid to policing the boundary, the less attention that can be paid to supervision on the premises. Implicitly, the choice of how to use \( q \) determines the level of both \( x \) and \( s \). Insofar as the rancher is concerned, the technically efficient choice is the one which maximizes \( x \). Indirectly, the rancher would maximize \( x \) with respect to \( x \). This does not imply that no attention is paid to policing the boundary (and certainly does not suggest that cattle are intentionally driven onto neighboring property). Following

\[
x_c(x) = \max_x x_c(x, y)
\]

with an \( x_c(y) \) determined as the solution. The culprit treats "x" as a free "input." Implicitly, this algorithm underlies the derivation of the usual culprit cost functions. Yet, if the law imposes any binding restriction upon \( s \), the "legally acceptable" production function shifts and Coase's reciprocity principle immediately follows. The law imposes harm upon the rancher, meaning either a lower output for any set of inputs, higher input requirements for any output, or higher costs if input prices do not change. Ceteris paribus, the legal requirement that \( s = 0 \) makes "land" more productive at the margin and "labor" less productive.

The solution to the bargaining problem is to allocate the aggregate resources in such a way to maximize joint profit. A formal statement of the problem requires a distinction between inputs acquired (denoted by \( \alpha \) and inputs utilized. For example, a farmer may rent \( z \) units of land, but it does not follow that \( z \) units of land are actually cultivated. Given this distinction, the problem may be tersely stated as:

\[
\pi(p, v, u, w) = \max_{v, u} \left[ p_x(x, u) + p_y(y, v) - w v \right]
\]

s. t. \( u \cdot v = 0 \)

where, by definition, \( v = \hat{e}_1 + \hat{e}_2 \), but \( \hat{e}_1 \neq v \) and \( \hat{e}_2 = v \). In contact, \( w \) is a fixed cost, so that the two firms maximize joint revenues by exchanging inputs. The product-specific inputs \((a, b)\) are not exchangeable, but \( z \) and \( q \) are. By mutual agreement, the legal rule is superceded, so that \( s \) adjusts implicitly.

For the mathematical purses, what is important is that the \( x(p, v, u) \) reduces to \( E(p, v, u) - w v \) and therefore "supports" the bargain (as in supporting hyperplanes, etc.). More importantly for interpretation, the solution yields a set of implicit input prices, including an implicit price of \( s \). If there is to be a long-run or competitive equilibrium (with joint profit equal to zero), the implicit input prices must equal the market input prices \((a)\) and the implicit price of \( s \) must equal zero. The fact that the implicit price of \( s \) has positive and negative components is irrelevant. Regardless of property rights, it is always possible for agents to adjust their initial positions so that both final outputs are increased by bargaining. More detail will be provided below, but the essential ingredients of a formal Coase Theorem may now be summarized:

(1) The prior choice of inputs, together with the prior set of legal rules, determine the initial position for the bargain.
(2) The legal rules pertaining to production are completely unrelated to the initial "endowments" that precede standard general equilibrium theory. The rules are "universal" — applied uniformly to all who choose to organize firms.
(3) So long as rational entrepreneurs correctly anticipate the outcome of the "bargain," neither party will allow itself to be exploited by the other.
The economic boundary between farmer and rancher is an independent decision regarding the use of (market) inputs. It is independent of legal entitlements.

A cooperative plan, which supersedes two independent production plans, determines the efficient distribution of inputs via-a-via the total land area.

The Topographical Diagrammaties

The mathematical fixation upon the (topological) properties of cost functions obscures the administrative functions of the firm, which pertain to the (topographical) questions about the actual use of property. To illustrate the difference, suppose the combined landholding (ze) of the rancher and farmer is the rectangular plot of land in Figure 1. To get to this position, each firm had to have acquired land (ze) and, following the usual discussions in the literature, we may assume that these acquisition decisions were made independently. There is a legal boundary somewhere in the diagram, but that boundary is merely recorded in the title company or in the terms of rental contracts. It has nothing whatsoever to do with how the property is utilized. Once a portion of the land (ze) is cultivated, an economic boundary appears, but that boundary does not, of itself, determine the allocation of resources and cattle via-a-via the land.

In principle, if cultivated land consists of homogeneous units, the efficient planning decision would be, insofar as it is possible, to distribute the seed input (b) uniformly over ze. However, cultivated land cannot be considered "homogeneous" if cattle stray onto a portion of it. With or without the cattle, the purpose of farm labor (q) is, along with other tasks, to distribute the seed according to the administrative plan. If there is to be such a "plan", it must be based upon some rational expectation with regard to the incidence (number and location) of straying cattle. Control of the cattle input (s) is the main purpose of the ranch labor (q) input. If ranch labor were perfectly abundant, the cattle, like the seed, could be uniformly distributed over grazing land and come to a screeching halt at the economic boundary.

The technological factor that lends novelty to the paradigm is the problem of controlling the movement of cattle. In context, there is no pure distinction between farm labor and ranch labor. Like the homogeneous land, there is one combined labor variable (q). Though the two individual firms each may have acquired labor (qz), independently, the economic problem is to allocate q efficiently to distribute inputs a and b efficiently over ze. The general nature of that distribution is illustrated in Figure 2. Some cattle (s) stray onto cultivated land, but common sense suggests a greater "cattle density" on grazing land. This allocation is not "unavoidable"; it is rather the result of a joint production plan. Similarly, there will be some crops planted and harvested in the "common" area, but farming activity will be less intensive there than in that portion of cultivated land that is cattle-free. Any attempt to model the specific distribution of inputs to specific points on the topographical map would engender endless complexity for no useful purpose. What is critical to Coase is that (1) the answers to these ethical
questions determine what the production functions are; (2) the answers will be different
under a cooperative plan than under two independent plans; and (3) although legal
rules must affect independent plans, they cannot affect cooperative plans. In simple
terms, so long as a farmer knows that his rights are not fully protected, he could (for
example) acquire less land and more labor, and the same amount of seed, knowing that
he can later bargain with the neighbor ("exchange" labor for land). In terms of Figures
1 and 2, the only things that change are the information that gets recorded in the title or
land rental contract and the "census" data that categorizes workers as "farm hands" and
"ranch hands".

V. PERSPECTIVES

The analytical foundation of this paper is the proposition that a technically efficient
production function is the solution to a complex mathematical problem and that the
structure of that problem is no different from that which underlies the derivation of cost
functions. In principle, technical efficiency relates to "things" that cannot or do not have
market prices. In Coase's terms, technical efficiency relates to "matters" that are less
costly to deal with administratively. The existence of generic transactions costs implies
that there is no clear dichotomy between the two problems. Although the absence of
transactions costs, together with a perfect price mechanism, restores the dichotomy,
these assumptions are insufficient to establish any clear delineation between the two
problems. Nevertheless, modeling choices are not wholly arbitrary. Rather, they must
be consistent with the nature of the problem under investigation. It is for this reason
that I insist on the inclusion of land and materials in \( x \).

In offering at least an outline of a mathematical Coase Theorem, which has defined
formal proof for more than thirty years, I suggest that the exercise is worthwhile.
However, its worth does lie in the results per se. Having worked through the logical
implications, it is probably best to follow Marshall and burn the math. In this instance,
however, the math is a necessary counterpoint to offset economic error that has been
disguised in math. Mindful of Marshall's caveat:

The chief use of pure mathematics in economic questions seems to be in helping a person to write down quickly, shortly and exactly, some of his
own thoughts for his own use... It seems doubtful whether anyone spends his time well in reading lengthy translations of economic
doctrines into mathematics, that have not been made by himself...[1920, x-xi]

I can only hope that the exercise is worthwhile to the reader, if only to foster a fuller
understanding of Coase's critique, his analytical point of view, and why his many articles
ultimately prove to be so thought-provoking.

In the sense that the exercise lends rigorous support to Coase's criticisms of
mechanical application of economic theory, it also lends rigorous support to the many
thoughtful nonmathematical discussions of the intangibles that exist in the neighbor-
hood of abstract dichotomies. Coase's intuitions with regard to these matters are as
exceptional as his inability or unwillingness to cope with the technical mathematical
detail. These intuitions and his intellectual leadership have proved far more effective
than any mathematical support could have been. Coase's contribution to economics is

that, to some extent, many applied theorists have come to think about what they are
thinking about. In the "long run", the mathematicians may follow suit.

Although this paper has been written in support of Coase, it need not and should not
be interpreted as an indictment of mathematical method, of the production-cost dichotomy,
or of governmental remedies designed to correct externalities. Above all, it is not
an indictment of Pigou. In fact, the chain of logical connections underlying the
paper has been laid out before in a virtually identical sequence — from Ch. II to Ch. IX
in The Economics of Welfare [1920]!

APPENDIX

NOTATIONAL CONVENTIONS

Allocative Variables

\[ x \]  "culprit" (rancher) output / any output in Sections I-II.
\[ y \]  "victim" (farmer) output.
\[ v \]  vector of market inputs.
\[ q \]  generic labor/capital input.
\[ s \]  generic land input.
\[ a \]  "culprit" material input (e.g., livestock).
\[ b \]  "victim" material input (e.g., seed).
\[ z \]  nuisance variable (e.g., straying cattle).
\[ n \]  number of "representative firms" in an industry.

Subscripts identify firm or industry — e.g., \( q \), \( q \), is labor allocated to ranching, \( x \), a vector of
inputs allocated to ranching, etc. Because ranchers do not use the \( b \) input,
\[ c_i = (q_i, z_i, a_i) \], and analogous for farmers who do not use input \( b \).

\*  Denotes equilibrium values of variables in Sections I-II,
\*  omitted in later sections.
\*  Denotes "acquired from factor market" to distinguish
from a firm's current stock of inputs and the manner
in which it chooses to employ them.

Prices, costs, etc.

\[ \rho \]  final output price.
\[ \omega \]  vector of factor prices.
\[ \sigma \]  total cost for a representative firm.
NOTES

1. The two classic articles are, of course, Cress (1937) and Cress (1960). The analytical part of the view is articulated in Cress (1968, Ch. 1). These themes are particularly evident in Cress (1948) and Cress (1958).

2. The usual textbook examples of mathematical production functions tend not to exhibit variable returns to scale because they are constructed to manipulate. An example of a two-input production function with variable returns to scale would be:

\[ y = \alpha x_1^{1/2} x_2^{1/2} \]

Given a fixed labor input, it follows that:

\[ \sum \partial y / \partial x_1 = \alpha \partial x_1 / \partial x_1 \]

where

\[ \sum \partial y / \partial x_1 = \alpha \partial x_1 / \partial x_1 \]

Suppose \( a = 0 \leq 0 \) and \( a = 0 \geq 0 \). Suppose \( a = 0 \leq 0 \) and \( a = 0 \geq 0 \). In light of the symmetry imposed on the function, the marginal firm would choose \( a = 0 \), only if the two inputs had the same price. Under a different set of input prices, factor combinations would adjust. Any number of exogenous input combinations also satisfy \( a = 0 \). In general, price is another unknown variable, but one which suggests trade-offs among imperfect systems.

3. The production function exhibits diminishing returns. In general, size is another unobservable variable, but one which suggests trade-offs among imperfect systems.

4. In reality, the core of the problem is much more complex than this and, to some extent, relates to statistical antecedents of cost. Of course, the more use of a production function implies that all outcomes are "intertwined." Nonetheless, a

"Mathematical" court would consider not only quantitative evidence such as how many, how long, how far, but also the subtleties such as what did the cattle do while they were grazing, were they under supervision, etc.

5. From this specific definition of the production function, an ordinary cost function, etc., is which reduces to if and is which reduces to if and is which reduces to if or if or if or if or if. However, this concept presumes that random (1) implement production plans independently (and cooperatively) and (2) have the right to maximize a welfare function for, whatever input contribution they choose to assemble. If held "sufficiently" diverse, these would emerge an explicit profit price of a and the cost function would shift.

6. Cress is not very explicit on these points, but he does hint at them. Whether the cattle raiser pays the farmer to leave his pastures of the land unutilized or himself rents the land by paying the landowner an amount slightly greater than the farmer would pay... The final result would be the same. (1960, 8)

The suggestion is that, if farmers' rights are not protected by the legal system, the burden of land acquisition shifts to the rancher. Cress (1968, 100) adds a new footnote to the reprint of the 1960 paper, in which he gives administrative decisions pertaining to partial abandonment of cultivation, selective planting, etc. The "exchange" is in perfect consonance with the modern legal-economic theory of contract... (1988) in the context, the authors suggest that the market could deal with such problems, at least administratively, without the existence of OSHA.

7. The cost controversy centered around Pigouv (1936, Book II, Ch. X) use of the term "increasing and decreasing supply price." Among the most insightful contributions to this controversy are Knight (1934),
UNDERSTANDING THE REMARKABLE SURVIVAL OF MULTIPLIER MODELS OF MONEY STOCK DETERMINATION

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INTRODUCTION

Assuming textbook authors reveal their intellectual and pedagogical preferences and beliefs, a careful survey of the leading intermediate textbooks in money and banking and macroeconomics reveals a uniform and virtually universal consensus—the multiplier model of money stock determination is widely viewed as the most appropriate and presumably most correct approach to the topic. In the leading seller by Mishkin [1992], for example, three chapters and a total of 64 pages (about 8½ percent of the text) are devoted to the development of the multiplier model. In justifying this extensive treatment, Mishkin argues “the complete model is the basis of much of the money supply analysis performed by practicing economists in the private sector and the government” (1992, 339). Since such consensus is not, in general, an enduring characteristic of monetary economics, one is tempted to “let sleeping dogs lie.” The problem is that the multiplier model, whether viewed from an analytical or empirical perspective, is at best a misleading and incomplete model and at worst a completely misspecified model.

The purpose of this paper is to assess the use and usefulness of the multiplier model from both the point of view of Federal Reserve policymaking, especially with regard to the operations of the Trading Desk, and with regard to ongoing theoretical and empirical work outside the Fed. The basic themes examined and developed can be simply stated: (1) ignoring various institutional and structural "details" has devastating implications for a large body of received theoretical and empirical work and the positive and normative economics which motivates and flows from it; and (2) the devastating implications relate mainly to the short-run (1-6 months) relationships among money, reserves (or the monetary base), and interest rates and do not necessarily contradict the proposition that, in general, the Fed, if it so chose, could control the growth of money within ±1½ percent range over a 6-12 month period.

DETERMINING VS. CONTROLLING VS. PREDICTING THE MONEY STOCK

Two familiar identities seem to dominate monetary economics: (1) \( MV = Y \) and (2) \( M = mB \), where \( M \) = the money stock (somewhat measured), \( V \) = velocity, \( Y \) = GNP, \( B \) = the monetary base, and \( m = \) the multiplier. Some have given new meaning to the term "reduced-form" by combining (1) and (2) into (3) \( mBV = Y \). This, along with the assumptions that \( m \) and \( V \) are "predictable" (i.e., stable stochastic processes), are orthogonal with respect to \( B \) and each other, and that \( B \) is controllable, implies that \( Y = (RB) \). Such expressions have provided the basis for a huge volume of empirical research, including the profession's recent infatuation with vector autoregressions, on such issues as the controllability of money, the relationship between the monetary and real sectors of the economy, the relationship between monetary policy and exchange